But sun it is not, when you say it is not;
And the moon changes, even as your mind.
What you will have it nam’d, even that it is;
And so it shall be so for Katharine.

The Taming of the Shrew

THE GEOMETER AS CONJUROR

The status of force in Élie Cartan's generally covariant theory of Newtonian gravity

Main question

- in Cartan's theory one is free to choose a flat geometry plus force, or a curved geometry without force
- *if our best theories*—Newtonian gravity remains pretty accurate on the whole—*tell us what there is, does gravitational force exist or doesn't it?*

Force

- is (gravitational) force a genuine feature of nature?
- or is it a possibly convenient but dispensable and ultimately empty mathematical construct?

Can theory determine ontology?

NO:

- the world and everything in it—small, large, distant, close—is there regardless of our attempts to represent or describe it, or to theorize about it
- ontology does not rest on theory; existence can be countenanced independently of our conceptual schemes

Can theory determine ontology?

YES:

- it makes no sense to countenance a kind of existence that goes beyond our conceptual schemes
- our best theories tell us what there is

PLAN

I. THEORY TELLS US WHAT THERE IS
II. CARTAN’S THEORY
III. DOES FORCE EXIST?
IV. TWO FREEDOMS
V. OTHER FORCES

I. THEORY TELLS US WHAT THERE IS

Frege, *Grundlagen der Arithmetik* (1884)

[…] der Satz, dass es kein rechtwinkliges, geradliniges, gleichseitiges Dreieck gebe, spricht eine Eigenschaft des Begriffes „rechtwinkliges, geradliniges, gleichseitiges Dreieck“ aus; diesem wird die Nullzahl beigelegt.

In dieser Beziehung hat die Existenz Aehnlichkeit mit der Zahl. Es ist ja Bejahung der Existenz nichts Anderes als Verneinung der Nullzahl.

Grundlagen der Arithmetik

Daher wäre es zuviel Behauptet, dass niemals aus den Merkmalen eines Begriffes auf die Einzigkeit oder Existenz geschlossen werden könne; […].

Es wäre auch falsch zu leugnen, dass Existenz und Einzigkeit jemals Merkmale von Begriffen sein könnten.

Meyerson, *Identité et réalité* (1907)

La science n'est pas *positive* et ne contient même pas de données positives, dans le sens précis qui a été donné à ce terme par Auguste Comte et ses sectateurs, de données « dépouillées de toute ontologie ». <u>L'ontologie fait corps avec la science elle-même et ne peut en être séparée</u>. Ceux qui prétendent l'en retrancher se servent inconsciemment d'un système métaphysique courant, d'un sens commun plus ou moins transformé par la science du passé qui leur est familière.

Cassirer, *Zur einstein’schen Relativitätstheorie* (1921)

Denn die erkenntnistheoretische Besinnung führt uns überall zu der Einsich, daß dasjenige, was die verschiedenen Wissenschaften den „Gegenstand“ nennen, kein ein für allemal Feststehendes, an sich Gegebenes ist, sondern daß es durch den jeweiligen Gesichtspunkt der Erkenntnis erst bestimmt wird. Je nach dem Wechsel dieses idellen Gesichtspunktes entstehen für das Denken verschiedene Klassen und verschiedene Systeme von Objekten.

Zur einstein'schen Relativitätstheorie

Immer gilt es daher in demjenigen, was die einzelnen Wissenschaften uns als ihre Objekte und „Dinge“ darbieten, die spezifischen logischen Bedingungen wiederzuerkennen, auf Grund deren sie festgestellt worden sind. Jede Wissenschaft *hat* ihren Gegenstand nur dadurch, daß sie ihn aus der gleichförmigen Masse des Gegebenen durch bestimmte Formbegriffe, die ihr eigentümlich sind, *heraushebt*.

1st sentence

- the *einzelne Wissenschaft* “Cartan theory” can *darbieten* force as an object or thing
- when it does, we *wiederzuerkennen* in that force the specific logical (*i.e*. geometrical) conditions (flat spacetime), *auf Grund deren* it was established
- force is *darbieten* by Cartan theory inasmuch as it is directly related to those *spezifischen logischen Bedingungen*

2nd sentence

- without theory, the world (‘*ur*-world’?) has the ‘pre-ontological’ character of a *gleichförmige Masse des Gegebenen*, a shapeless soup, too messy and unintelligible to constitute a legitimate ontology
- the resources of theory are required to give it shape and form; to render it manageable, manipulable; to turn it into a genuine ontology

Two kinds of existence

- one can distinguish between a ‘strong’ existence which presupposes a minimum of order, form, intelligibility, manageability; and a weaker, more general kind that doesn’t
- some would then claim that only the stronger kind is of any (scientific?) interest; that the weaker doesn’t matter
- but should existence depend on ‘user-friendliness’?

Cassirer, *Zur einsteinschen Relativitätstheorie*

Was immer diese Gegenständlichkeit bedeuten mag, in keinem Falle kann sie mit dem zusammenfallen, was die naive Weltansicht als die Wirklichkeit *ihrer* Dinge, als die Wirklichkeit der sinnlichen Wahrnehmungsobjekte anzusehen pflegt. Denn von *dieser* Wirklicheit sind die Objekte, von denen die wissenschaftliche Physik handelt, und für die sie ihre Gesetze aufstellt, schon durch ihre allgemeine Grundform geschieden. Daß Begriffe, wie die der Masse und der Kraft, des Atoms oder des Äthers, des magnetischen oder elektrischen Potentials, ja Begriffe wie die des Drucks oder der Temperatur, keine einfachen Dingbegriffe, keine Nachbildungen konkreter in der Wahrnehmung gegebener Einzelinhalte sind: das bedarf nach allem, was die Erkenntnistheorie der Physik selbst über den Sinn und Ursprung dieser Begriffe festgestellt hat, kaum der Erörterung mehr. Was wir in ihnen besitzen, sind ersichtlich nicht Reproduktionen einfacher Ding- oder Empfindungsinhalte, sondern theoretische Setzungen und Konstruktionen, die darauf gerichtet sind, das bloß Empfindbare in ein Meßbares und damit erst in einen „Gegenstand der Physik“, d. h. in einen solchen *für* die Physik zu verwandeln.

Quine, “Notes on existence and necessity” (1940)

The ontology to which one’s use of language commits him comprises simply the objects that he treats as falling […] within the range of values of his variables.

“On what there is” (1948)

Now how are we to adjudicate among rival ontologies? Certainly the answer is not provided by the semantical formula “To be is to be the value of a variable”; this formula serves rather, conversely, in testing the conformity of a given remark or doctrine to a prior ontological standard. We look to bound variables in connection with ontology not in order to know what there is, but in order to know what a given remark or doctrine […] *says* there is; and this much is quite properly a problem involving language. But what there is is another question.

“On what there is”

It is no wonder, then, that ontological controversy should tend into controversy over language. But we must not jump to the conclusion that what there is depends on words. Translatability of a question into semantical terms is no indication that the question is linguistic. To see Naples is to bear a name which, when prefixed to the words ‘sees Naples’, yields a true sentence; still there is nothing linguistic about seeing Naples.

“On what there is”

Our acceptance of an ontology is, I think, similar in principle to our acceptance of a scientific theory […]: we adopt […] the simplest conceptual scheme into which the disordered fragments of raw experience can be fitted and arranged. Our ontology is determined once we have fixed upon the over-all conceptual scheme which is to accomodate science in the broadest sense […]. To whatever extent the adoption of any system of scientific theory may be said to be a matter of language, the same […] may be said of the adoption of an ontology.

Word & object (1960)

[…] I look to variables and quantification for evidence as to what a theory says that there is, not for evidence as to what there is; but the point can be missed […].

Word & object

In § 49 I spoke of dodges whereby philosophers have thought to enjoy the systematic benefits of abstract objects without suffering the objects. There is one more such dodge in what I have been inveighing against in these last pages: the suggestion that the acceptance of such objects is a linguistic convention distinct somehow from serious views about reality.

Analytic & synthetic

- blurring distinction between analytic & synthetic makes difference between theoretical formalism and ontology less clear too

II. CARTAN’S THEORY

Why Cartan’s theory?

- rare example of total empirical equivalence, along with nontrivial theoretical difference
- one can view it as one theory that comes in different versions
- gauge theory?
- two different freedoms

Empirical equivalence

- hard to find genuine empirical equivalence elsewhere
- subtle difference can usually be found
- perhaps a conjectural difference, yet to be manifested
- Bohmian mechanics and standard QM
- Aharonov-Bohm effect (reveals vector potential?)

Space⊕time

Space⊕time is represented by the 4-dimensional differential manifold M. The level surfaces of the time function $t: M \to \mathbb{R}$ (or x^0) are the 3-dimensional simultaneity surfaces S_t. The three spatial coordinates x^i satisfy

$$\mathrm{d}t(\partial/\partial x^i) = \frac{\partial t}{\partial x^i} = 0,$$

$i = 1, 2, 3.$

Affine structure

A covariant derivative D, with components

$$\Gamma^i_{jk} = \mathrm{d}x^i(D_{\partial/\partial x^j}\,\partial\big/\partial x^k),$$

will provide a generally covariant notion of straightness, and indeed of inertial motion, through the equation $a = D_{\dot\sigma}\dot\sigma = \mathbf{0}$, in other words

$$a^i = \mathrm{d}x^i(a) = \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k$$

$$= \tfrac{d}{dt}\mathrm{d}x^i(\dot\sigma) + \Gamma^i_{jk}\mathrm{d}x^j(\dot\sigma)\mathrm{d}x^k(\dot\sigma) = 0,$$

where σ is a worldline and $\dot\sigma$ its tangent (and $\mathrm{d}t(\dot\sigma) = 1$).

Separation of space and time

Each tangent space $T_x M$ can be broken up $T_x M_0 \oplus T_x M_s$ $[T_x M_0 = (T_x M)_0, T_x M_s = (T_x M)_s]$ into a 1-dimensional temporal subspace $T_x M_0$ and a 3-dimensional spatial subspace $T_x M_s$, so that any tangent vector $v \in T_x M$ can be expressed as the direct sum $v = v_0 \oplus v_s$ of a temporal part $v_0 = P_0 v \in T_x M_0$ and spatial part $v_s = P_s v \in T_x M_s$, where P_0 and P_s are projection operators.

Space$\oplus$time metric

The space$\oplus$time metric $g: T_x M \times T_x M \to \mathbb{R}$ can likewise be broken up $g = g_0 \oplus g_s$ into a temporal metric $g_0 : T_x M_0 \times T_x M_0 \to \mathbb{R}$ and a spatial metric $g_s : T_x M_s \times T_x M_s \to \mathbb{R}$, where $g(u,v) = g_0(u_0, v_0) + g_s(u_s, v_s)$ and $u = u_0 \oplus u_s,\ v = v_0 \oplus v_s$.

Matrix elements

The matrix elements

$$g_{0i} = g_{i0} = g(\partial/\partial x^0, \partial/\partial x^i) = g(\partial/\partial t, \partial/\partial x^i)$$

vanish for the spatial values $1, 2, 3$ of the index i.

Compatibility

$D_X g$ has to vanish along all directions X

Orthogonality

The temporal metric is the tensor product $g_0 = \mathrm{d}t \otimes \mathrm{d}t$. Since a spatial vector $v_s \in T_x M_s$ joins simultaneous events we can say that v_s (or $T_x M_s$ or S_t) is orthogonal to $\mathrm{d}t$ (or to g_0), for $\mathrm{d}t(v_s)$ and

$$g_0(v_s, v_s) = g_0 \oplus 0_s(\mathbf{0} \oplus v_s, \mathbf{0} \oplus v_s) = g_0(\mathbf{0}, \mathbf{0}) + 0_s(v_s, v_s)$$

both vanish, where $0_s : T_x M_s \times T_x M_s \to \mathbb{R}$ is the null spatial metric.

More orthogonality

A temporal vector $v \in T_x M_0$ (spatial projection $P_s v = 0$) tangent to M at point $x = (t, r)$ ($r \in \mathrm{S}_t$) is likewise orthogonal to S_t since

$$g_s(v_0, v_0) = 0_0 \oplus g_s(v_0 \oplus \mathbf{0}, v_0 \oplus \mathbf{0}) = 0_0(v_0, v_0) + g_s(\mathbf{0}, \mathbf{0})$$

also vanishes, where

$$0_0 = \bar{\mathbf{0}} \otimes \bar{\mathbf{0}} : T_x M_0 \times T_x M_0 \to \mathbb{R}$$

is the null temporal metric (the covector $\bar{\mathbf{0}}$ can be seen as the vanishing differential of a constant function).

Rigging

Since the temporal subspace $T_x M_0$ is a ray, all vectors in it will be multiples of the unique normalized vector satisfying $\mathrm{d}t(v) = 1$. The normalized temporal vector at every point provides a vector field tangent to a congruence or ‘rigging’ allowing spatial positions to be identified over time.

Gradient and differential

We obtain

$$g_s^{\sharp}(\cdot): T_x^* M_s \to T_x M_s$$

by partial evaluation of

$$g_s(\cdot,\cdot): T_x M_s \times T_x M_s \to \mathbb{R}.$$

It can be used to produce the gradient

$$\nabla\phi = (\mathrm{d}\phi)^{\sharp} = g_s^{\sharp}(\mathrm{d}\phi) \in T_x M_s$$

of the function $\phi: \mathrm{S}_t \to \mathbb{R}$ from the differential $\mathrm{d}\phi: T_x M_s \to \mathbb{R}.$

Force and potential

Force is the negative gradient

$$F = -\nabla\phi \in T_x M_s$$

of the potential ϕ satisfying Poisson's equation

$$\nabla^2\phi = \rho,$$

where ρ is the mass density.

Affine freedom 1 (‘whim’)

The motion of a body satisfies

$$F = -\nabla\phi = D_{\dot\sigma}\dot\sigma.$$

We can replace D with another derivative operator D' and write

$$D'_{\dot\sigma}\dot\sigma = \mathbf{0} \Leftrightarrow D_{\dot\sigma}\dot\sigma = -\nabla\phi.$$

Affine freedom 2 (‘MFU’)

In a matter-filled universe we are even free to add to ϕ a field ψ satisfying $\nabla^2\psi = 0$ which, through the potential

$$\bar{\phi} = \phi + \psi$$

determines a new derivative operator $\bar{D}$ satisfying

$$\bar{D}_{\dot{\sigma}}\dot{\sigma} = -\nabla\bar{\phi} = D_{\dot{\sigma}}\dot{\sigma} - \nabla\psi = -\nabla(\phi + \psi).$$

Island universe

- no freedom to vary the potential if the mass density vanishes outside a certain region

III. DOES FORCE EXIST?

1 (realism?)

- theory determines ontology
- so force *really* vanishes and reappears—exists or not—at the geometer's whim
- peculiar, fragile kind of existence
- some would rather associate more stability, solidity with existence
- unsatisfactory that existence should be up to the geometer

2 (instrumentalism?)

- the statements and objects of theory are not to be taken seriously, literally
- force is just empty mathematics
- so who cares if it can be conjured up at the stroke of a pencil
- virtual, mathematical legerdemain
- never mind about the world

3 (conventionalism?)

- ontology does matter, but isn’t determined by theory
- force exists (or not) in the world regardless of what theory says
- its disappearance into geometry is confusing mathematical artifice
- buried, misleadingly concealed, but still there
- Poincaré of *La science et l’hypothèse* (1902)?

4

- a criterion is chosen that privileges one of the apparently equivalent descriptions
- status of force is determined by that description
- criterion could be historical (*e.g. Newton had force*)
- could be along lines of simplicity, elegance, economy, parsimony
- for Poincaré *flat geometry + force* preferable as simpler
- Poincaré of *La valeur de la science* (1905)

5 (consilience?)

- appeal to other theories
- description containing force privileged because supported by other theories
- force—the entity, not just the word—being present elsewhere, we know it exists
- might as well choose a description consistent with other theories
- Whewell’s *consilience of inductions*; Friedman

Universal freedom (Hepburn)

- but what if force can be geometrized away in all theories?
- freedom to dispense with force could be universal
- of what help are other theories then?

IV. TWO FREEDOMS

What are the two freedoms?

1. *diffeomorphic* (or *coordinate*) freedom
2. *affine* freedom

Existence and invariance

- (objective) existence can be associated, even identified, with invariance (Cassirer, Meyerson, Weyl, Nozick; Earman?)
- the presence (or absence) of force is given invariant or generally covariant expression in Cartan’s theory
- so force is there (or not) for all observers, or rather for all coordinate systems

‘Coordinate’ invariance

For instance if the components

$$a^i = \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k$$

all vanish, they will vanish with respect to another coordinate system $\overline{x}^i$ too:

$$\overline{a}^i = \ddot{\overline{x}}^i + \overline{\Gamma}^i_{jk}\dot{\overline{x}}^j\dot{\overline{x}}^k = 0.$$

Opinion

Quantities like $\mathrm{d}x^i(v) \neq \mathrm{d}\overline{x}^i(v)$ and

$$\Gamma^i_{jk} = \mathrm{d}x^i\left(D_{\partial/\partial x^j}\,\partial/\partial x^k\right)$$

$$\neq \overline{\Gamma}^i_{jk} = \mathrm{d}\overline{x}^i\left(D_{\partial/\partial \overline{x}^j}\,\partial/\partial \overline{x}^k\right)$$

(where $v \in T_xM;\ \mathrm{d}x^i, \mathrm{d}\overline{x}^i \in T_x^*M$) are subjective;

they vary according to changes

$$x^i \leftrightarrow \overline{x}^i$$

of ‘point of view.’

Affine freedom

- should all that is subject to affine freedom be accordingly dismissed as unreal, as mere opinion?
- does affine freedom have invariants? what are they? are they then the fundamental reality?
- or is affine freedom fundamentally different from diffeomorphic freedom?

Differences between freedoms

- diffeomorphisms form group, with identifiable invariants
- can affine freedom be characterized by a group?
- coordinate systems represent *physical* ‘points of view’
- affine freedom reflects geometer’s whim

V. OTHER FORCES

Loss of generality?

- gravity is discreet and inconspicuous in its operation
- so we’re not too alarmed if told that gravitational force doesn’t exist, or that it amounts to no more than geometry
- but not all forces are so unobtrusive
- what about rocket propulsion?

Rocket propulsion

- if acceleration due to rocket propulsion is absorbed into new affine structure, has force disappeared?
- still flames & smoke
- if we switch off propulsion, the new affine structure sees acceleration and force

Equilibrium

- just after liftoff, gravity-propulsion equilibrium
- fire & smoke *vs*. geometry
- mismatch

Reduction

- all forces ultimately geometrizable by reduction to manifestly geometrizable elementary forces?