A priori prejudice in Weyl's unintended unification of gravitation and electricity

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Outline

Introduction

Claim

- Weyl's TGE (theory of gravitation and electricity) came out of 'mathematical balance,' or perhaps 'mathematical justice': out of the **equal rights of direction and length**

Why bother with Weyl's theory?

- wasn't the theory wrong, and notoriously refuted by Einstein?

- whatever its **direct** relationship with experience, the TGE has been anything but empirically sterile, having led to today's gauge theories, whose great empirical success we know about

- not even mathematically fruitless: active fields of mathematical research (Weyl manifolds *etc.*) are closely related to the TGE

- the TGE also provides an interesting example of how directly an elaborate theory can emerge from simple prejudice

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Serendipity

Unintended unification

- it is almost always claimed (by Folland, Trautman, Perlick, Vizgin and others) that Weyl **deliberately** unified gravitation and electricity

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“Gravitation und Elektrizität”

Sitzungsber. d. K. Preuß. Akad. d. Wissenschaften 1918, p.465

Indem man die erwähnte Inkonsequenz beseitigt, kommt eine Geometrie zustande, die überraschenderweise, auf die Welt angewendet, **nicht nur die Gravitationserscheinungen, sondern auch die des elektromagnetischen Feldes erklärt**.

Weyl to Einstein, 10 December 1918

Übrigens müssen Sie nicht glauben, daß ich von der Physik her dazu gekommen bin, neben der quadratische noch die lineare Differentialform in die Geometrie einzuführen; sondern ich wollte wirklich diese “Inkonsequenz,” die mir schon immer ein Dorn im Auge gewesen war, endlich einmal beseitigen und bemerkte dann zu meinem eigenen Erstaunen: das sieht so aus, als erklärt es die Elektrizität.

Two candidates

- so the unification was unexpected, and came out of *a priori* considerations

- but there are two possibilities:

  1. mathematical justice: the equal rights of direction and length (ERDL)
  2. the 'infinitesimal agenda' Ryckman traces back to Husserl

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Infinitesimal agenda

- the infinitesimal agenda appears, in the 'context of discovery,' as no more than a handful of vague and unmotivated intimations
- passages (next two screens) that give substance to the agenda by providing epistemological foundations and motivation belong to a subsequent 'context of justification': 1926, 1931

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Philosophie der Mathematik und Naturwissenschaft
Munich, Oldenbourg, '2000' ($\geq$ 1926), p.173

Erkennt man neben dem physischen einen **Anschauungsraum** an und behauptet von ihm, daß seine Maßstruktur aus Wesensgründen die euklidischen Gesetze erfülle, so steht dies mit der Physik nicht in Widerspruch, sofern sie an der euklidischen Beschaffenheit der **unendlich kleinen Umgebung** eines punktes O (in dem sich das Ich momentan befindet) festhält [. . .]. Aber man muß dann zugeben, daß die Beziehung des Anschauungsraumes auf den physischen um so vager wird, je weiter man sich vom Ichzentrum entfernt. Er ist einer Tangentenebene zu vergleichen, die im Punkte O an eine krumme Fläche, den physischen Raum, gelegt ist: in der unmittelbaren Umgebung von O decken sich beide, aber je weiter man sich von O entfernt, um so willkürlicher wird die Fortsetzung dieser Deckbeziehung zu einer eindeutigen Korrespondenz zwischen Ebene und Fläche.

“Geometrie und Physik”
Die Naturwissenschaften **19**, 1931, p.52

Die Philosophen mögen recht haben, daß unser Anschauungsraum, gleichgültig, was die physikalische Erfahrung sagt, euklidische Struktur trägt. Nur bestehe ich allerdings dann darauf, daß zu diesem Anschauungsraum das Ich-Zentrum gehört und daß die Koinzidenz, die Beziehung des Anschauungsraumes auf den physischen um so vager wird, je weiter man sich vom Ich-Zentrum entfernt. In der theoretischen Konstruktion spiegelt sich das wider in dem Verhältnis zwischen der krummen Fläche und ihrer Tangentenebene im Punkte P: beide decken sich in der unmittelbaren Umgebung des Zentrums P, aber je weiter man sich von P entfernt, um so willkürlicher wird die Fortsetzung dieser Deckbeziehung zu einer eindeutigen Korrespondenz zwischen Fläche und Ebene.

Equal rights

Why favour the ERDL?

- the ‘local’ epistemology or ‘telescepticism’ that Ryckman traces back to Husserl could be more of an outgrowth of the theory than its roots
- the ‘infinitesimal agenda’ is too vague and insubstantial, as it appears in 1918 (a few unmotivated hints and adumbrations), to be logically sufficient
- surprisingly, mathematical justice **is logically sufficient**: electromagnetism (away from charges—up to Hodge duality at any rate) comes, through a handful of natural and simple steps, straight out of the ERDL
- the formal development is manifestly guided, as we shall see, by the ERDL
- furthermore the **parallel between coordinates and gauge**, which Weyl draws over and over (next four screens), can be interpreted as a parallel between direction and length

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"Gravitation und Elektrizität"

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Die auftretenden Formeln müssen dementsprechend eine doppelte Invarianzeigenschaft besitzen: 1. sie müssen **invariant** sein **gegenüber beliebigen stetigen Koordinatentransformationen**, 2. sie müssen ungeändert bleiben, **wenn man die g_{ik} durch λg_{ik} ersetzt**, wo λ eine willkürliche stetige Ortsfunktion ist.

"Reine Infinitesimalgeometrie"
Mathematische Zeitschrift **2**, 1918, p.396

Zum Zwecke der analytischen Darstellung denken wir uns 1. ein bestimmtes Koordinatensystem und 2. den an jeder Stelle willkürlich zu wählenden Proportionalitätsfaktor im skalaren Produkt festgelegt; damit ist ein "**Bezugssystem**"[9] für die analytische Darstellung gewonnen. [. . .]

⋮

9. Ich unterscheide also zwischen "Koordinatensystem" und "Bezugssystem."

“Reine Infinitesimalgeometrie”
Mathematische Zeitschrift **2**, 1918, p.398

In alle Größen oder Beziehungen, welche metrische Verhältnisse analytisch darstellen, müssen demnach die Funktionen g_{ik}, φ_i in solcher Weise eingehen, daß Invarianz stattfindet 1. gegenüber einer beliebigen Koordinatentransformation (“Koordinaten-Invarianz”) und 2. gegenüber der Ersetzung von (7) durch (8) (“Maßstab-Invarianz”).

“Eine neue Erweiterung der Relativitätstheorie

Annalen der Physik **59**, 1919, p.101

Um den physikalischen Zustand der Welt an einer Weltstelle durch Zahlen charakterisieren zu können, muß 1. die Umgebung dieser Stelle auf **Koordinaten** bezogen sein und müssen 2. gewisse **Maßeinheiten** festgelegt werden. Die bisherige Einsteinsche Relativitätstheorie bezieht sich nur auf den ersten Punkt, die Willkürlichkeit des Koordinatensystems; doch gilt es, eine ebenso prinzipielle Stellungnahme zu dem zweiten Punkt, der Willkürlichkeit der Maßeinheit, zu gewinnen.

Coordinates up to gauge

- the parallel between coordinates and gauge, which Weyl draws over and over, can be seen as a parallel between direction and length

- coordinates, up to gauge, give only direction

- a system x^a assigns to every point $P \in M$ a basis $\partial_a \in T_P M$, and a dual basis

$$dx^a = g^\flat(\partial_a) = g(\partial_a, \cdot\,) \in T_P^* M$$

which provides the components $V^a = \langle dx^a, V \rangle$ of every vector $V \in T_P M$; $a = 0, \ldots, 3$

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Coordinates and direction

- a gauge transformation $g \mapsto e^{2\lambda}g$ induces a transformation $V \mapsto e^{\lambda}V$, or $V^a \mapsto e^{\lambda}V^a$, through

$$e^{2\lambda}g(V,V) = g(e^{\lambda}V, e^{\lambda}V) = \sum_{ab} g(e^{\lambda}\partial_a, e^{\lambda}\partial_b)V^aV^b = \sum_{ab} g(\partial_a,\partial_b)e^{\lambda}V^ae^{\lambda}V^b$$

- direction, given by the ratios

$$e^{\lambda}V^0 : e^{\lambda}V^1 : e^{\lambda}V^2 : e^{\lambda}V^3 = V^0 : V^1 : V^2 : V^3,$$

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Analogy

DIRECTION	**LENGTH**
coordinates (up to gauge)	gauge
parallel transport	congruent transport
gravitation	electricity
Levi-Civita connection Γ^a_{bc}	length connection A
directional curvature R^a_{bcd} (of Γ^a_{bc})	length curvature $F = dA$
geodesic coordinates y^a (at P): $\Gamma^a_{bc} = 0$	geodesic gauge (at P) $A' = A + d\lambda = 0$
equivalence principle: $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c \mapsto \ddot{y}^a$	equivalence principle: $\alpha = -lA \mapsto \alpha' = 0$

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“Nozione di parallelismo in una varietà qualunque . . . ”
Rendiconti del Circolo matematico di Palermo **42**, 1917, p.173

L’incontro, anzi il maneggio continuativo di tali simboli [di Riemann] in questioni di così alto interesse generale mi ha condotto a ricercare se non sia possibile ridurre alquanto l’apparato formale che serve abitualmente ad introdurli e a stabilirne il comportamento covariante. Un perfezionamento in proposito è effettivamente possibile [. . .] venne via via ampliandosi per far debito posto anche all’interpretazione geometrica. In sulle prime avevo creduto di trovarla senz’altro nei lavori originali di Riemann [. . .]; ma ce n’è appena un embrione. [. . .] D’altra parte non c’è traccia [. . .] di quelle specificazioni (nozione di direzioni parallele in una varietà qualunque [. . .]) che riconosceremo indispensabili dal punto di vista geometrico.

“Nozione di parallelismo in una varietà qualunque . . . ”
Rendiconti del Circolo matematico di Palermo **42**, 1917, p.174

[. . .] cercando di caratterizzare il parallelismo di due direzioni (α), (α') uscenti da due punti vicinissimi P e P'. All’uopo si ricorda che qualunque varietà V_n si può risguardare immersa in uno spazio euclideo S_n a un numero abbastanza elevato N di dimensioni, e si rileva anzitutto che, immaginando spiccata da P una generica direzione (f) di S_n, il parallelismo ordinario in tale spazio richiederebbe

$$\text{angolo}\widehat{(f)(\alpha)} = \text{angolo}\widehat{(f)(\alpha')},$$

per qualunque (f). Orbene, il parallelismo in V_n si definisce, limitandosi ad esigere che la condizione sia soddisfatta **per tutte le (f) appartenenti a** V_n (ossia alla giacitura di S_N tangente in P a V_n).

“Nozione di parallelismo in una varietà qualunque ...”
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[seguito] A giustificazione di tale definizione va notato che, mentre essa riproduce, come è necessario, il comportamento per le V_n euclidee, ha in ogni caso carattere intrinseco, perché in definitiva risulta dipendente soltanto dalla metrica di V_n, e non anche dall'ausiliario spazio ambiente S_N. Infatti la traduzione analitica della nostra definizione di parallelismo si concreta come segue: Riferita la V_n a coordinate generali x_i $(i = 1, 2, \ldots, n)$, siano dx_i gli incrementi corrispondenti al passaggio da P a P'; $\xi^{(i)}$ i parametri spettanti a una generica direzione (α) uscente da P; $\xi^{(i)} + d\xi^{(i)}$ quelli spettanti ad una direzione infinitamente vicina (α'), spiccata da P'. La condizione di parallelismo è espressa dalle n equazioni

$$d\xi^{(i)} + \sum_{j,l=1}^{n} \begin{Bmatrix} jl \\ i \end{Bmatrix} dx_j \, \xi^{(l)} = 0 \tag{1}$$

$(i = 1, 2, \ldots, n)$, designando $\begin{Bmatrix} jl \\ i \end{Bmatrix}$ i noti simboli di Christoffel.

“Nozione di parallelismo in una varietà qualunque ...”
Rendiconti del Circolo matematico di Palermo **42**, 1917, p.174

[seguito] Una volta acquisita la legge con cui si passa da un punto a un punto infinitamente vicino, si è senz’altro in grado di eseguire il trasporto di direzioni parallele lungo una qualsiasi curva C. Se $x_i = x_i(s)$ ne costituiscono le equazioni parametriche, basta evidentemente risguardare, nelle (1), le x_i e subordinatamente le $\left\{ {jl \atop i} \right\}$ come funzioni assegnate, le $\xi^{(i)}$ come funzioni da determinarsi del parametro s, e si ha il sistema lineare ordinario

$$\frac{d\xi^{(i)}}{ds} + \sum_{j,l=1}^{n} \left\{ {jl \atop i} \right\} \frac{dx_j}{ds} \xi^{(l)} = 0$$

$(i = 1, 2, \ldots, n)$, riducibile ad una forma tipica [...].

“Nozione di parallelismo in una varietà qualunque …”

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[…] Ecco qualche conseguenza geometrica.

1° La direzione parallela in un punto generico P ad una direzione (α) uscente da un altro punto qualsiasi P_0 dipende in generale dal cammino secondo cui si passa da P_0 a P. L’indipendenza dal cammino è proprietà esclusiva delle varietà euclidee.

Connections

Affinely connected manifold

- Weyl calls the manifold M **affinely connected** if every tangent space $T_P M$ ($P \in M$) is connected to all the neighbouring tangent spaces $T_{P'} M$ through a map

$$\mathfrak{T}_X : T_P M \to T_{P'} M : V_P \mapsto V_{P'} = \mathfrak{T}_X V_P$$

which is linear both in the 'main' argument $V_P \in T_P M$ and in the (short) directional argument

$$X = P' - P,$$

where P' (being near P) and hence X are seen as belonging to $T_P M$

- being linear, $\mathfrak{T}_X$ will be represented by a matrix, indeed by

$$\mathfrak{T}^a_c = \langle dx^a, \mathfrak{T}_X \partial_c \rangle = \sum_b \mathfrak{T}^a_{bc} X^b = \sum_b \langle dx^a, \mathfrak{T}_{\partial_b} \partial_c \rangle \langle dx^b, X \rangle$$

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Component differences

- Weyl refers specifically to the components

$$\delta V^a = \langle dx^a_{P'}, V_{P'} \rangle - \langle dx^a_P, V_P \rangle,$$

requiring them to be linear in the components

$$X^b \text{ and } V^c_P = \langle dx^c_P, V_P \rangle$$

- the bilinear function

$$\Gamma^a(\{X^b\}, \{V^c\}) = \delta V^a$$

will be a matrix, represented by Γ^a_{bc}

- so the difference δV^a is

$$-\sum_{bc} \Gamma^a_{bc} X^b V^c$$

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Geodesic coordinates

- with respect to the geodesic coordinates y^a which make

$$\Gamma^a_c = \sum_b \Gamma^a_{bc} X^b = \langle dy^a, \nabla_X \partial_{c(y)} \rangle \text{ and } \delta V^a$$

vanish, leaving the components V^a unchanged, $\mathfrak{T}^a_c$ becomes the identity matrix

$$\delta^a_c = \langle dy^a, \mathfrak{T}_X \partial_{c(y)} \rangle \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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From direction to length

- with mathematical justice in mind Weyl turns to length, using the same scheme

- to clarify his argument we can take just a single component of the difference

$$\{\delta V^0, \ldots, \delta V^3\},$$

calling it δl (this will be the 'squared-length-difference scalar')

- the upper (or 'image') index of Γ^a_{bc} disappears, leaving

$$\delta l = \sum_{bc} \Gamma_{bc} V^b X^c$$

(one can perhaps view this hybrid, intermediate connection Γ_{bc} as something like $\langle A, \nabla_{\partial_b} \partial_c \rangle$)

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(one can perhaps view this hybrid, intermediate connection Γ_{bc} as something like $\langle A, \nabla_{\partial_b} \partial_c \rangle$)

From direction to length

- with mathematical justice in mind Weyl turns to length, using the same scheme
- to clarify his argument we can take just a single component of the difference

$$\{\delta V^0, \ldots, \delta V^3\},$$

calling it δl (this will be the 'squared-length-difference scalar')

- the upper (or 'image') index of Γ^a_{bc} disappears, leaving

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The length connection

- if we now take a single component of the main argument $\{V^0, \ldots, V^3\}$, calling it l (this will be the squared length), the second lower (or 'main argument') index of Γ_{bc} disappears as well, leaving

$$\delta l = \sum_c \Gamma_c l X^c,$$

where $\Gamma_c = \langle A, \partial_c \rangle$ are the components of a one-form, denoted A with electricity in mind ($c = 0, \ldots, 3$)

Weyl's requirements

- but this isn't exactly Weyl's argument, which is better conveyed as follows
- the length connection A generating the squared-length-difference scalar δl has to be linear in the squared length l and the direction X
- a linear function $A(l, X) = \delta l$ of a scalar l and a vector X yielding a scalar δl will be a one-form:

$$\delta l = \langle \alpha, X \rangle = -l \langle A, X \rangle,$$

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- Weyl in fact writes

$$dl = -l d\varphi$$

(the misleading d's should not be understood globally—or even locally, in the TGE; for $F = d^2\varphi$ will become the Faraday two-form and d^2 vanishes)

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Exactness

- an **exact** one-form $A = d\mu$ would make congruent transport integrable, removing from the 'dilation'

$$e^{\int_\gamma A} = e^{\int d\mu} = e^{\Delta\mu}$$

  all dependence on the path $\gamma : [0, 1] \to M$, where $\Delta\mu = \mu_1 - \mu_0$ is the difference between the values $\mu_1 = \mu(P_1)$ and $\mu_0 = \mu(P_0)$ of μ at $P_1 = \gamma(1)$ and $P_0 = \gamma(0)$

- but integrability is precisely what Weyl wanted to give up, for the equal rights of direction and length

- an exact one-form, even if it dilates, would not preclude remote (integrable) comparisons, leaving the fundamental injustice unresolved

- justice demands that the curl $F = dA$ of the one-form A not vanish everywhere

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Geodesic gauge

- Weyl's requirement that the squared-length-difference one-form

$$\alpha = -lA$$

be locally eliminable by recalibration confirms that A is a possibly inexact one-form

- l does not generally vanish, so Weyl's requirement means that $A + d\lambda$ must vanish at a point, where the gauge λ is **geodesic**

- since $d\lambda$ is a one-form, A will be one too

- while $d\lambda$ is exact, Weyl only asks that it cancel A **at a point**, so A does not even have to be exact (or even closed for that matter)

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Raum Zeit Materie
Berlin, Springer, 1923, p.122

Ein Punkt P hängt also mit seiner Umgebung metrisch zusammen, wenn von jeder Strecke in P feststeht, welche Strecke aus ihr durch kongruente Verpflanzung von P nach dem beliebigen zu P unendlich benachbarten Punkte P' hervorgeht. Die einzige Forderung, welche wir an diesen Begriff stellen (zugleich die weitgehendste, die überhaupt möglich ist), ist diese: Die Umgebung von P läßt sich so eichen, daß die Maßzahl einer jeden Strecke in P durch kongruente Verpflanzung nach den unendlich benachbarten Punkten keine Änderung erleidet.

But A is a tensor!

- one may wonder how the tensor A can correspond to the connection Γ^a_{bc}, which is not a tensor

- the components $A_a = \langle A, \partial_a \rangle = \Gamma_a$ only transform 'tensorially' with respect to coordinate transformations

$$A_a \mapsto \overline{A}_b = A_a \langle d\overline{x}^b, \partial_{a(x)} \rangle$$

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$$A_a \mapsto A'_a = A_a + \partial_a \lambda$$

the components A_a do not transform like a tensor, and can be cancelled, for instance

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Electricity

Maxwell's equations

- with $F = dA$ and its consequence $dF = 0$ before him, Weyl couldn't help seeing the electromagnetic 4-potential A, the Faraday 2-form $F = dA$, and Maxwell's two homogeneous equations expressed by $dF = 0$, in other words

$$\nabla \cdot B = 0 \quad \text{and} \quad \nabla \times E + \partial B/\partial t = 0$$

  (not to mention an electromagnetic 'equivalence principle,' according to which the squared-length-difference scalar δl and one-form α, as well as the electromagnetic 4-potential A, can be generated or eliminated at a point by a suitable gauge λ)

In coordinates

- in coordinates one would write:

$$F_{ab} = F(\partial_a, \partial_b) = \partial_a A_b - \partial_b A_a \leftrightarrow \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

  where $A_a = A(\partial_a)$; E_x, E_y, E_z are the components of the electric field and B_x, B_y, B_z those of the magnetic field

- or

$$F = \frac{1}{2} \sum_{ab} F_{ab}\, dx^a \wedge dx^b$$

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Electromagnetism away from sources

- Maxwell's two homogeneous equations are given by the vanishing three-form

$$dF = \frac{1}{2}\sum_{bc} dF_{bc} \wedge dx^b \wedge dx^c = \frac{1}{6}\sum_{abc} \partial_a F_{bc}\, dx^a \wedge dx^b \wedge dx^c = 0$$

with components $dF(\partial_a, \partial_b, \partial_c) = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}$

- Maxwell's other two equations are obtained, in 'source-free' form, by setting d^*F equal to zero, where $^*F$ is the Hodge dual of the Faraday two-form

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“Reine Infinitesimalgeometrie”

Mathematische Zeitschrift **2**, 1918, p.385

Nach dieser Theorie ist **alles Wirkliche, das in der Welt vorhanden ist, Manifestation der Weltmetrik**; die physikalischen Begriffe sind keine andern als die geometrischen. [...][1]

⋮

1. Ich bin verwegen genug, zu glauben, daß die Gesamtheit der physikalischen Erscheinungen sich aus einem einzigen universellen Weltgesetz von höchster mathematischer Einfachheit herleiten läßt.

Compensation

Gauge freedom

- since only the curl $F = dA$ 'counts,' there is freedom to add the differential $d\mu$ of a function μ to A

- by transforming the four-potential according to

$$A \rightarrow A' = A + d\mu,$$

the four-curl

$$F = dA' = d(A + d\mu) = dA + d^2\mu = dA$$

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Further 'dilation'

- even if the curl unaffected by the differential $d\mu$, length changes

- transporting X_0 from point P_0 with value $\mu_0 = \mu(P_0)$ to point P_1 with value $\mu_1 = \mu(P_1)$, the final squared length $g_1(X_1, X_1)$ acquires the additional (integrable) factor $e^{\Delta\mu}$

- for the function μ dilates according to

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$$g_1(X_1, X_1) = e^{\int_\sigma A} g_0(X_0, X_0) \neq e^{\int_\sigma A'} g_0(X_0, X_0)$$

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Conformal transformation

- to re-establish the invariance of length we have to compensate multiplying the metric by the conformal factor e^{μ}:

$$g \rightarrow g' = e^{\mu} g$$

- together the two transformations leave length unchanged:

$$g'_1(X_1, X_1) = e^{\mu_1} g_1(X_1, X_1) = e^{\int_\sigma A'} g'_0(X_0, X_0) = e^{\int_\sigma A} e^{\Delta\mu} e^{\mu_0} g_0(X_0, X_0)$$

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‘Metric’ connection

- the relationship between the above two transformations is also expressed by **Weyl compatibility**

- the metric g is (strictly) **compatible** with the connection ∇ if

$$\nabla g = 0$$

- in that case the straightest geodesics (satisfying $\nabla_{\dot{\sigma}}\dot{\sigma} = 0$) will also be ‘stationary,’ satisfying

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Einstein

Einstein's objection

- out of a sense of mathematical justice, then, Weyl made congruent displacement just as path-dependent as parallel transport

- but experience, objected Einstein, is unfair, showing congruent dispacement to be **integrable**: clocks running at the same rate at one point will, he argued, **continue** to run at the same rate at another point, however they get there (whatever the requirements of mathematical justice!)

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Einstein to Weyl, 15 April 1918

So schön Ihre Gedanke ist, muss ich doch offen sagen, dass es nach meiner Ansicht ausgeschlossen ist, dass die Theorie die Natur entspricht. Das ds selbst hat nämlich reale Bedeutung. Denken Sie sich zwei Uhren, die relativ zueinander ruhend neben einander gleich rasch gehen. Werden sie voneinander getrennt, in beliebiger Weise bewegt und dann wieder zusammen gebracht, so werden sie wieder gleich (rasch) gehen, d. h. ihr relativer Gang hängt nicht von der Vorgeschichte ab. Denke ich mir zwei Punkte P_1 & P_2 die durch eine Zeitartige Linie verbunden werden können. Die an P_1 & P_2 anliegenden zeitartigen Elemente ds_1 und ds_2 können dann durch mehrere zeitartigen Linien verbunden werden, auf denen sie liegen. Auf diesen laufende Uhren werden ein Verhältnis $ds_1 : ds_2$ liefern, welches von der Wahl der verbindenden Kurven unabhängig ist.—Lässt man den Zusammenhang des ds mit Massstab- und Uhr-Messungen fallen, so verliert die Rel. Theorie überhaupt ihre empirische Basis.

Clean spectral lines

- four days later Einstein reformulated the objection in terms of the 'proper frequencies' of atoms (rather than genuine macroscopic clocks) "of the same sort"

- if such frequencies depended on the (electromagnetic) vicissitudes of the atoms—and hence on the worldlines followed—the chemical elements the atoms would make up if brought together would not have the clean spectral lines one sees

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Einstein to Weyl, 19 April 1918

[. . .] wenn die Länge eines Einheitsmassstabes (bezw. die Gang-Geschwindigkeit einer Einheitsuhr) von der Vorgeschichte abhingen. Wäre dies in der Natur wirklich so, dann könnte es nicht chemische Elemente mit Spektrallinien von bestimmter Frequenz geben, sondern es müsste die relative Frequenz zweier (räumlich benachbarter) Atome der gleichen Art im Allgemeinen verschieden sein. Da dies nicht der Fall ist, scheint mir die Grundhypothese der Theorie leider nicht annehmbar, deren Tiefe und Kühnheit aber jeden Leser mit Bewunderung erfüllen muss.

Dead end?

- but even if experience shows congruent displacement to be integrable, it may be wrong to say that the equal rights of direction and length led nowhere

- the structure that came out of Weyl's ostensibly groundless sense of mathematical justice would survive in our standard gauge theories, whose accuracy is much less questionable

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- the structure that came out of Weyl's ostensibly groundless sense of mathematical justice would survive in our standard gauge theories, whose accuracy is much less questionable

Final remarks

Levels of experience

There are various levels of ‘experience,’ ranging from the most concrete to the most abstract: from the most obvious experimental level, having to do with the results of particular experiments, to principles, perhaps even instincts, distilled from a lifetime of experience. One such principle could be Einstein’s “I am convinced that He does not play dice,” to which, having (say) noticed that the causal regularities behind apparent randomness eventually tend to emerge, he may ultimately have been led by experience: by his own direct experience, together with his general knowledge of science and the world. One would nonetheless hesitate to view so general and abstract a principle as being *a posteriori*, empirical. It is clearly not *a posteriori* with respect to any particular experiment; only, if at all, with respect to a very loose, general and subjective kind of ongoing experience, capable of being processed and interpreted in very different ways.

Empirical roots

An unexpected empirical fertility of apparently *a priori* and unempirical prejudice can sometimes be accounted for in terms of a derivation, however indirect, from experience: by attributing remote empirical roots to principles which at first seem to have nothing at all to do with experience. Fair enough—the world can be experienced in very different ways, some much less obvious and straightforward than others. But here we have a prejudice which, however subtle and developed Weyl's faculties for processing experience, seems to be completely unempirical. Perhaps the shortcomings of the theory are best blamed, then, on the manifestly unempirical nature of the prejudice from which it was derived.

Progeny

But what about the great empirical success of the progeny, of the gauge theories that would follow? Fluke? Irrelevant? Are the descendants 'illegitimate,' and not so direct after all? Is the connection between today's gauge theories and the ERDL too tenuous to be worth speaking of? The scheme of compensation outlined above remains unmistakably present in today's gauge theories, and is central to their success …