IF BERTLMANN HAD THREE FEET

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It is argued that perfect quantum correlations cannot be due to additive conservation.

> Dr. Bertlmann likes to wear two socks of different colours. Which colour he will have on a given foot on a given day is quite unpredictable. But when you see that the first sock is pink you can already be sure that the second sock will not be pink. Observation of the first, and experience of Bertlmann, gives immediate information about the second. [1]

Most interesting features of quantum mechanics have to do with coherence (in other words with interference, with phase), which will not, however, be at issue here at all. Coherence is brought out with respect to different bases, but here the same (product) basis is adhered to throughout.

It is often claimed that *conservation accounts for quantum correlations* (by which *perfect* quantum correlations will be meant). The underlying intuition is well expressed by Bertlmann's socks, or by the fact that the distribution of wine over two glasses can be worked out—provided one knows the total amount in both—by a measurement on one of them. Or consider a conservative classical Hamiltonian $H = T + V(q)$, where T is kinetic energy and the potential V depends only on position. Conservation means that exchanges of kinetic and potential energy along a trajectory satisfy $H_0 = T + V$, where H_0 is the total energy of the motion. Kinetic energy will then be a function only of position, so that at any stage of the motion $T(q) = H_0 - V(q)$ can be deduced from the potential; the two energies are perfectly correlated. Or take two free classical particles, each one subject only to the influence of the other, with initial momenta p_0 and p'_0. Even if they collide their total momentum will remain $\pi = p_0 + p'_0$; the momentum $p' = \pi - p$ of the primed particle can always be derived from the momentum p of the other. Such instances of additive conservation are paradigmatic.

Quantum correlations are similar, especially at a given instant, and with only two subsystems; but they have nothing to do with conservation. When the contrary is claimed it seems that *additive* conservation is meant; but that can be broken up into two logically independent parts: 1. *conservation*; and 2. *an* *'additivity' condition*, presently to be defined and denoted (λ). Quantum correlations can have nothing to do with time, which has everything to do with conservation; so what is fundamentally at issue is additivity.

I will argue that an additivity condition can be constructed to account for quantum correlations with two subsystems, *but only with two*; where there are more, *quantum correlations are too strong to be explained by additivity*. An explanation that only works in a restricted special case should be viewed as no explanation at all; so quantum correlations have nothing to do with additivity.

Take three socks (on an equal number of feet) rather than two: once the pink sock is found on one foot, we know the remaining socks are on the other feet, but we cannot infer where the blue one is. With three glasses a measurement on one glass only tells us how much wine is in the other two together, not how much is in the third. Triorthogonal decompositions appear to go beyond the knowledge available in the above cases, and indeed to tell us where the blue sock is, or how much wine is in the third glass.

Consider the triorthogonal decomposition*

$$|\Psi\rangle = \sum_m c_m |\alpha_m^1\rangle |\alpha_m^2\rangle |\alpha_m^3\rangle \in \mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \mathcal{H}^3, \tag{1}$$

where the Hilbert spaces $\mathcal{H}^r = \text{span}\{|\alpha_1^r\rangle, |\alpha_2^r\rangle, \ldots\}$ ("span" denotes the *closed* span) have the same dimensionality, and $\langle \alpha_i^r | \alpha_j^r \rangle = \delta_{ij}$ $(r = 1,2,3)$. The state† $|\Psi\rangle$ determines a *trijective* or one-to-one-to-one correspondence $|\alpha_m^1\rangle \leftrightarrow |\alpha_m^2\rangle \leftrightarrow |\alpha_m^3\rangle$ correlating the bases $\{|\alpha_m^1\rangle\}$, $\{|\alpha_m^2\rangle\}$, and $\{|\alpha_m^3\rangle\}$, $m = 1,2,\ldots$. To make the correspondence observable and give rise to correlations, we can construct the self-adjoint operator

$$\mathbf{A} = \mathbf{A}^1 + \mathbf{A}^2 + \mathbf{A}^3 : \mathcal{H} \to \mathcal{H},$$

where

$$\begin{aligned} \mathbf{A}^1 &= A^1 \otimes I \otimes I : \mathcal{H} \to \mathcal{H} \\ \mathbf{A}^2 &= I \otimes A^2 \otimes I : \mathcal{H} \to \mathcal{H} \\ \mathbf{A}^3 &= I \otimes I \otimes A^3 : \mathcal{H} \to \mathcal{H}, \end{aligned}$$

and the three (maximal) operators A^r have the form

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* According to Schmidt's theorem (Schmidt 1907), every vector in the tensor product of two Hilbert spaces can be given a biorthogonal decomposition (see also von Neumann 1932 pp.228-32, and Schrödinger 1935). But *almost no* (see Clifton 1994) vectors in the tensor product of three Hilbert spaces admit a triorthogonal decomposition (see also Peres 1995).

† Which is considered primitive or 'given'; its physical origins—who knows what conditions may have produced it—are not at issue.

$$\sum_m \lambda_m^r \,|\,\alpha_m^r\rangle\langle\alpha_m^r\,| : \mathcal{H}^r \to \mathcal{H}^r,$$

$r = 1,2,3.$ The operator A^r establishes a one-to-one correspondence $\lambda_1^r \leftrightarrow |\,\alpha_1^r\rangle,$ $\lambda_2^r \leftrightarrow |\,\alpha_2^r\rangle,$ $\lambda_3^r \leftrightarrow |\,\alpha_3^r\rangle, \ldots$ between eigenvalues and basis vectors, thus extending to the three spectra $\Lambda^r = \{\lambda_1^r, \lambda_2^r, \ldots\}$ the aforementioned trijective correspondence between the bases (again $r = 1,2,3$). The discovery of an eigenvalue therefore selects one in both of the other two factor spaces. This will be particularly surprising if we require that

$$\lambda_m^1 + \lambda_m^2 + \lambda_m^3 = \lambda \qquad (\lambda)$$

for all m (so that $\mathbf{A}\,|\,\Psi\rangle = \lambda\,|\,\Psi\rangle$); for then the entire system possesses an amount λ of the physical quantity $\mathfrak{A}$ represented by $\mathbf{A}$, whose exact distribution over all three subsystems would be determined by a measurement on any one of them. We expect this with two subsystems, maybe not with three.

Consider the Cartesian product $\Lambda = \Lambda^2 \times \Lambda^2 \times \Lambda^3 = \{(\lambda^1_{m^1}, \lambda^2_{m^2}, \lambda^3_{m^3})\}$ of the spectra, and the subset

$$\Lambda_{(\lambda)} = \{(\lambda^1_{m^1}, \lambda^2_{m^2}, \lambda^3_{m^3}) : \lambda^1_{m^1} + \lambda^2_{m^2} + \lambda^3_{m^3} = \lambda\} \subset \Lambda$$

satisfying condition (λ). The discovery of an eigenvalue λ_n^s (the value $s = 1,2$ *or* 3 of the superscript is chosen by the experimenter, that of the subscript by nature) will determine a subset

$$\Lambda_{(\lambda; m^s = n)} = \{(\lambda^1_{m^1}, \lambda^2_{m^2}, \lambda^3_{m^3}) : \lambda^1_{m^1} + \lambda^2_{m^2} + \lambda^3_{m^3} = \lambda; m^s = n\} \subset \Lambda_{(\lambda)},$$

which would be a singleton if there were only two subsystems.

The triorthogonal decomposition (1) determines another subset of $\Lambda_{(\lambda)}$, namely $\Lambda_{(1)} = \{(\lambda_m^1, \lambda_m^2, \lambda_m^3)\} \subset \Lambda_{(\lambda)}$. Here the discovery of the same eigenvalue λ_n^s would select the triple

$$\Lambda_{(1; m^s = n)} = (\lambda_n^1, \lambda_n^2, \lambda_n^3) \subset \Lambda_{(\lambda; m^s = n)}.$$

We would have $\Lambda_{(1)} = \Lambda_{(\lambda)}$ and $\Lambda_{(1; m^s = n)} = \Lambda_{(\lambda; m^s = n)}$ with two subsystems, *and only then*. This means that the correlations due to the triorthogonal decomposition, being stronger than those due to (λ), cannot be attributed to such an additivity condition.

The matter can also be seen as follows. A vector $|\Phi\rangle$ belonging to eigenspaces corresponding to eigenvalues $\lambda^r_{m^r}$ that add up to λ will be an eigenvalue of $\mathbf{A}$ corresponding to λ; in other words conditions

$$\left(|\alpha^1_{m^1}\rangle\langle\alpha^1_{m^1}|\otimes|\alpha^2_{m^2}\rangle\langle\alpha^2_{m^2}|\otimes|\alpha^3_{m^3}\rangle\langle\alpha^3_{m^3}|\right)|\Phi\rangle = |\Phi\rangle \text{ and } \lambda^1_m + \lambda^2_m + \lambda^3_m = \lambda$$

together imply that $\mathbf{A}|\Phi\rangle = \lambda|\Phi\rangle$. So

$$\Omega_{(\lambda)} = \text{span}\{|\alpha^1_{m^1}\rangle|\alpha^2_{m^2}\rangle|\alpha^3_{m^3}\rangle : \lambda^1_{m^1} + \lambda^2_{m^2} + \lambda^3_{m^3} = \lambda\}$$

is the eigenspace belonging to λ.

The commuting set $\{\mathbf{A}, \mathbf{A}^s\}$ (again $s = 1, 2$ *or* 3), representing the amount of $\mathfrak{A}$ respectively possessed by the whole system and by subsystem s, *will not be complete*, since $\dim \Omega_{(\lambda;\, m^s=n)} > 1$ for the subspace

$$\Omega_{(\lambda; m^s=n)} = \text{span}\{|\alpha^1_{m^1}\rangle|\alpha^2_{m^2}\rangle|\alpha^3_{m^3}\rangle : \lambda^1_{m^1} + \lambda^2_{m^2} + \lambda^3_{m^3} = \lambda;\, m^s = n\}$$

determined by λ^s_n. But with a triorthogonal expansion, the two measurements $\mathbf{A}$ and $\mathbf{A}^s$ determine a single product $|\alpha^1_m\rangle|\alpha^2_m\rangle|\alpha^3_m\rangle$. So the correlations contained in a triorthogonal expansion go beyond those due to the set $\{\mathbf{A}, \mathbf{A}^s\}$, which would otherwise only have selected the larger subspace $\Omega_{(\lambda; m^s=n)}$.

If there were only two subsystems, $\{\mathbf{A}, \mathbf{A}^s\}$ would be a *complete* commuting set, since a single product would be determined by measurement of $\mathbf{A}$ and $\mathbf{A}^s$; $\{\mathbf{A}, \mathbf{A}^s\}$ would represent neither more nor less correlation than what is contained in a triorthogonal decomposition.

Time has so far been out of the picture; but one can also wonder about evolution. The triorthogonal decomposition is preserved if $|\alpha^1_m\rangle|\alpha^2_m\rangle|\alpha^3_m\rangle$ $(m = 1, 2, \ldots)$ are energy eigenvectors, for then the time evolution operator does not change their directions. We can then speak of *conservation*, and say that the correlations in question *cannot be attributed to an additive conservation law*.

References

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